EFFECT OF THE NONLINEAR DEPENDENCE OF SURFACE TENSION ON TEMPERATURE ON THE SHAPE OF A FREE SURFACE WITH CONVECTIVE MOVEMENT IN A LIQUID

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A considerable number of works have been devoted to the effect of capillary forces on liquid equilibrium and movement under conditions close to weightlessness. It should be noted that with normal gravitation when the specific liquid surface is large, capillary forces may become decisive. To a considerable extent the diversity of capillary forces is due to the dependence of surface tension on temperature and the concentration of surface-active substances.

Recently, several works have appeared [1-3] in which the anomalous dependence of surface tension coefficient on temperature is considered:

$$\sigma_1 = \sigma_{01} + \alpha_1(T' - T^*)^2,\ \sigma_{01} = \text{const},\ \alpha_1 = \text{const}. \quad (1)$$

This preceded a series of experiments [4, 5] which revealed the existence of a dependence $\sigma = \sigma_1(T')$ for a broad class of substances, e.g., for aqueous solutions of high molecular alcohols, some binary metal alloys, and nematic liquid crystals.

This work is a theoretical consideration of nonstationary thermocapillary (TC) and thermogravitation (TG) convection in a thin layer of viscous incompressible liquid for the nonlinear dependence of surface tension coefficient on temperature according to rule (1). The flow which arises is compared with movement characteristics when the surface tension coefficient depends on temperature according to a linear rule

$$\sigma_2 = \sigma_{20} - \alpha_2 T',\ \sigma_{20} = \text{const},\ \alpha_2 = \text{const},\ \alpha_2 > 0. \quad (2)$$

Statement of the Problem. In the initial instant of time a radiation pulse (e.g., a laser pulse) passes through a surface within a liquid of thickness H occupying a round cylindrical cell of radius R ($R \gg H$). Due to absorption of the radiation within the volume of the liquid in the track, a region of higher temperature arises with a maximum located at the surface of the liquid, and heating decreases quite rapidly from the center. Energy distribution within the radiation beam is assumed to be Gaussian. In the following instants of time the heat source does not operate. The bottom, side walls of the cell, and the free liquid surface are thermally insulated. It is assumed that the duration of the laser pulse is short compared with the typical time of convective heat and mass transfer. For example, in an experiment in [6] the pulse duration was 10^{-7} sec. Consequently, in the initial instant of time in the vessel there is equilibrium temperature distribution

$$T' = T_0 + (T_1 - T_0)\exp(-(r'/a')^2 + \alpha(z' - H)). \quad (3)$$

Here T_0 is temperature of the whole liquid before the start of irradiation; T_1 is maximum temperature in the heated region; a' is radiation beam radius; α is liquid radiation absorption factor. The thermal nonuniformity formed causes TC and TG convection in the layer. It is assumed that changes of liquid physical parameters as a result of a change in temperature are negligibly small, with the exception of a change in density and surface tension coefficients.

Taking account of these assumptions mathematical formulation of the problem includes equations: Navier–Stokes, thermal conductivity, and continuity described in Boussinesq approximations.

In view of the fact that in this work the change in liquid layer thickness $h' = h'(r', t')$ is considered, we shall assume that

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Moscow. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 38-47, March-April, 1992. Original article submitted May 15, 1990; revision submitted January 25, 1991.

$$p' = p_0 + \rho g(h'(r', t') - z') + p''$$

[$p'' = p''(r', z', t')$ is deviation of pressure from hydrostatic, p_0 is external pressure, ρ is original density].

In a cylindrical coordinate system the equations have the form

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial r'} + v'\frac{\partial u'}{\partial z'} = -g\frac{\partial h'}{\partial r'} - \frac{1}{\rho}\frac{\partial p''}{\partial r'} + \nu\nabla^2 u',$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial r'} + v'\frac{\partial v'}{\partial z'} = -\frac{1}{\rho}\frac{\partial p''}{\partial z'} + g\beta(T' - T_0) + \nu\nabla^2 v',$$

$$\frac{\partial T'}{\partial t'} + u'\frac{\partial T'}{\partial r'} + v'\frac{\partial T'}{\partial z'} = \varkappa\nabla^2 T',$$

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r'u') + \frac{\partial v'}{\partial z'} = 0.$$

Boundary conditions at the side walls and the bottom are attachment conditions for the liquid:

$$u' = v' = 0,\ z' = 0,\ r' = R.$$

There are symmetry conditions at the cell axis:

$$u' = \partial v'/\partial r' = 0,\ r' = 0.$$

At the free surface [7] with $z' = h'(r', t')$

$$(p' - p_0)\, n_i = \sigma_{ik} n_k + \sigma n_i\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \tau_k \frac{\partial \sigma}{\partial x_k}\tau_i,\ i = 1, 3,\ k = 1, 3, \tag{4}$$

where σ_{ik} is viscous stress tensor; $\boldsymbol{\tau}$ is tangent to the liquid surface; $\mathbf{n}$ is the normal directed into the liquid; $1/R_1 + 1/R_2$ is liquid surface curvature.

By using known mathematical expressions for surface curvature and the normal to it we obtain from Eq. (4) in projections on axes r' and z' the following expressions:

$$\begin{aligned} p''\dot{h}' &= 2\rho\nu\dot{h}'\frac{\partial u'}{\partial r'} - \rho\nu\left(\frac{\partial v'}{\partial r'} + \frac{\partial u'}{\partial z'}\right) - \frac{\sigma\dot{h}'}{\sqrt{1+(\dot{h}')^2}}\left(\frac{\ddot{h}'}{1+(\dot{h}')^2} + \frac{\dot{h}'}{r'}\right) + \\ &\quad + \left(\frac{\partial\sigma}{\partial r'} + \dot{h}'\frac{\partial\sigma}{\partial z'}\right)\frac{1}{\sqrt{1+(\dot{h}')^2}}, \\ p'' &= 2\rho\nu\frac{\partial v'}{\partial z'} - \rho\nu\dot{h}'\left(\frac{\partial v'}{\partial r'} + \frac{\partial u'}{\partial z'}\right) - \frac{\sigma}{\sqrt{1+(\dot{h}')^2}}\left(\frac{\ddot{h}'}{1+(\dot{h}')^2} + \frac{\dot{h}'}{r'}\right) - \\ &\quad - \left(\frac{\partial\sigma}{\partial r'} + \dot{h}'\frac{\partial\sigma}{\partial z'}\right)\frac{\dot{h}'}{\sqrt{1+(\dot{h}')^2}}. \end{aligned} \tag{5}$$

Here $\dot{h}' = \partial h'/\partial r'$, $\ddot{h}' = \partial^2 h'/\partial r'^2$. At the surface the vertical component of liquid velocity equals the velocity of the change $h'(r', t')$: $\partial h'/\partial t' + u'\partial h'/\partial r' = v'$. We also impose a symmetry condition on the shape of the surface $\partial h'/\partial r' = 0$, $r' = 0$. At the side surface of the cell we prescribe a condition for damping of movement $h' = H$, $\partial h'/\partial r' = 0$, $r' = R$.

The initial conditions are a condition for liquid immobility $u' = v' = 0$, a flat surface $h' = H$, and prescribed temperature distribution (3). Dimensional values are used in the equations: u' and v' are radial and vertical components of velocity, ν and κ are coefficients of viscosity and thermal diffusivity.

Let L be the characteristic dimension of a region in which liquid movement is localized in the radial direction. Then the fact that it will be localized is known from numerical calculations in [8] and an experiment [6]. Dimension L is governed by the radius of a circle outside which surface tension is almost unchanged and the liquid does not move. The radius of the light spot heating the liquid with t = 0 is less than or of the order of L, $a' \lesssim L$.

In view of absence of liquid movement with $r' > L$ transfer of heat in this region may only occur as a result of thermal conductivity.

Then we shall consider a thin layer of liquid: $\varepsilon = H/L \ll 1$. In the approximation of the thin layer radial heat transfer due to thermal conductivity is insignificant. Then boundary conditions with $r' \to L$ take the form

$$\partial T'/\partial r' = 0,\ u' = v' = 0,\ h' = H,\ \partial h'/\partial r' = 0.$$

At the bottom and free surface boundary conditions for temperature are written as

$$\partial T'/\partial z' = 0,\ z' = 0,$$
$$\partial T'/\partial z' + \dot{h}'\partial T'/\partial r' = 0,\ z' = h'(r',\ t').$$

A relationship between velocities $v' \sim \varepsilon u'$ follows from the continuity equation.

We estimate radial velocity from Eq. (5) assuming that the surface is not curved. Then

$$\rho\nu\left(\frac{\partial u'}{\partial z'} + \frac{\partial v'}{\partial r'}\right) \approx \frac{\partial \sigma}{\partial r'} = \frac{d\sigma}{dT'}\frac{\partial T'}{\partial r'},$$

whence for linear relationship (2) characteristic velocity $u_{02} = \alpha_2 \Delta T H/\rho\nu L$. In the case of nonlinear relationship (1) $d\sigma/dT' = 2\alpha_1(T' - T^*)$, temperature T^* corresponding to the minimum surface tension changes within the limits $T_0 \le T^* \le T_1$. In estimates we shall assume that $d\sigma_1/dT' \sim -2\alpha_1\Delta T/2 = -\alpha_1\Delta T$. The sign is selected so that with $T' < T^*$ the direction of liquid movement at the surface corresponds to the case of a linear relationship $\sigma = \sigma_2(T')$. Then $u_{01} = \alpha_1(\Delta T)^2 H/\rho\nu L$.

The ratio of gravitation and thermocapillary forces is specified by the Bond number Bd. For a nonlinear dependence $\sigma_1(T')$

$$\mathrm{Bd} = \frac{\rho g \beta H^2}{\alpha_1 \Delta T} = \frac{\rho g \beta \Delta T H}{\alpha_1 (\Delta T)^2/H} = \frac{p_g''}{p_{th}''}.$$

In this work the case of $\mathrm{Bd} \sim 1$ is considered, consequently, $p_g'' \sim p_{th}''$, and as a scale for pressure it is possible to take any of these values. Let $p_{ch} = p_{th}'' = \alpha_1(\Delta T)^2/H$.

Taking account of the estimates made we write dimensionless variables:

$$r = r'/L,\ a = a'/L,\ z = z'/H,\ h = h'/H,$$
$$u = u'/u_0,\ v = v'/(\varepsilon u_0),\ p = p''/p_{ch},$$
$$\theta = (T' - T_0)/\Delta T,\ \theta^* = (T^* - T_0)/\Delta T,\ t = t'/\tau,\ \Delta T = T_1 - T_0.$$

We introduce the time scale later.

In dimensionless variables equations and boundary conditions have the form

$$\frac{H^2}{\nu\tau}\frac{\partial u}{\partial t} + \varepsilon^2 \mathrm{Re}\left(u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial r} - \frac{\mathrm{Bd}}{\beta_T}\frac{\partial h}{\partial r} + \varepsilon^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2}\right) + \frac{\partial^2 u}{\partial z^2},$$
$$\frac{\varepsilon^2 H^2}{\nu\tau}\frac{\partial v}{\partial t} + \varepsilon^4 \mathrm{Re}\left(u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mathrm{Bd}\,\theta + \varepsilon^4\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \varepsilon^2\frac{\partial^2 v}{\partial z^2}, \tag{6}$$
$$\frac{\mathrm{Pr}\,H^2}{\nu\tau}\frac{\partial \theta}{\partial t} + \left(u\frac{\partial \theta}{\partial r} + v\frac{\partial \theta}{\partial z}\right)\varepsilon^2 \mathrm{Pr}\,\mathrm{Re} = \varepsilon^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) +$$
$$+ \frac{\partial^2 \theta}{\partial z^2},\ \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0.$$

Here $\mathrm{Pr} = \nu/\kappa$; $\mathrm{Re} = u_0 L/\nu$; $\beta_T = \beta\Delta T$. There are initial conditions with $t = 0$:

$$u = v = 0,\ h = 1,\ \theta = \exp(-r^2/a^2 + \alpha H(z - 1)), \tag{7}$$

and boundary conditions:

$$u = v = 0,\ \partial\theta/\partial z = 0,\quad z = 0;$$
$$u = v = 0,\ \partial\theta/\partial r = 0,\quad r = 1;\tag{8}$$
$$u = \partial v/\partial r = 0,\qquad r = 0.$$

At the free boundary with z = h(r, t)

$$p\dot{h} + \frac{\partial u}{\partial z} + \varepsilon^2\left(\frac{\partial v}{\partial r} - 2\dot{h}\frac{\partial u}{\partial r}\right) + \frac{\sigma_{10}}{\alpha_1(\Delta T)^2}\frac{\varepsilon^2\dot{h}}{\sqrt{1+(\varepsilon\dot{h})^2}}\left(\frac{\ddot{h}}{1+(\dot{h})^2\varepsilon^2} + \frac{\dot{h}}{r}\right) +$$
$$+ 2(\theta^* - \theta)\left(\frac{\partial\theta}{\partial r} + \dot{h}\frac{\partial\theta}{\partial z}\right)\frac{1}{\sqrt{1+\varepsilon^2(\dot{h})^2}} = 0,$$
$$p = \varepsilon^2\left(2\frac{\partial v}{\partial z} - \dot{h}\left(\varepsilon^2\frac{\partial v}{\partial r} + \frac{\partial u}{\partial z}\right)\right) - \tag{9}$$
$$- \frac{\varepsilon^2\sigma_{10}}{\alpha_1(\Delta T)^2}\frac{1}{\sqrt{1+\varepsilon^2(\dot{h})^2}}\left(\frac{\ddot{h}}{1+\varepsilon^2(\dot{h})^2} + \frac{\dot{h}}{r}\right) +$$
$$+ 2(\theta^* - \theta)\left(\frac{\partial\theta}{\partial r} + \dot{h}\frac{\partial\theta}{\partial z}\right)\frac{\varepsilon^2\dot{h}}{\sqrt{1+\varepsilon^2(\dot{h})^2}},\ \frac{\partial\theta}{\partial z} + \varepsilon^2\dot{h}\frac{\partial\theta}{\partial r} = 0,$$
$$\frac{L}{u_0\tau}\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r} = v;\ \frac{\partial h}{\partial r} = 0,\ r = 0;\ h = 1,\ \frac{\partial h}{\partial r} = 0,\ r = 1.$$

It can be seen from Eqs. (6)-(9) that in the nonstationary problem in question there are several time scales connected with development of velocity and temperature fields, and with a change in layer thickness.

We introduce the time scales: $\tau_1 = H^2/\nu$ is the time of the nonstationary process of liquid movement, $\tau_2 = L/u_0$ is typical time for a change in surface shape, $\tau_3 = \Pr H^2/\nu$ is time for development of the temperature field. We assume that Bd ~ 1, Re ~ 1, Pr ~ 1, $\sigma_{01}/(\alpha_1(\Delta T)^2) \sim 1$. Then, $\tau_1 \sim \tau_3$, $\tau_2/\tau_1 = 1/\varepsilon^2\mathrm{Re}$.

Case of Short Times. We assume that $\tau = \tau_1$. By ignoring in Eq. (6) terms of the order of ε^2 and higher, we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z^2} = -\frac{\partial p}{\partial r} - \frac{\mathrm{Bd}}{\beta_T}\frac{\partial h}{\partial r},$$
$$\frac{\partial p}{\partial z} = \mathrm{Bd}\,\theta,\ \Pr\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2},\ \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0.\tag{10}$$

At the free surface instead of (9) we have

$$\frac{\partial h}{\partial t} = 0,\ p = 0,\ \frac{\partial\theta}{\partial z} = 0,\ \frac{\partial u}{\partial z} = 2(\theta - \theta^*)\frac{\partial\theta}{\partial r}.\tag{11}$$

Initial conditions (7) and remaining boundary conditions (8) are unchanged.

From the first equation in (11) we find that h = 1, i.e., at a given stage the shape is unchanged. Then in the first equation of (10) we drop the term $(\mathrm{Bd}/\beta_T)\partial h/\partial r$. By solving Eqs. (10) successively by the Fourier method we obtain expressions in the form of series for temperature, pressure, and velocity. The expression for temperature is used subsequently with asymptotic combination of solutions in the case of short and long times

$$\theta = \exp\left(-\frac{r^2}{a^2}\right)\left\{\frac{1-\exp(-\alpha H)}{\alpha H} + \right.$$
$$\left. + \sum_{k=0}^{\infty}\frac{2\left((-1)^k - \exp(-\alpha H)\right)\exp\left(-(\pi k)^2 t/\Pr\right)}{\alpha H\left(1+(\pi k/(\alpha H))^2\right)}\cos\pi kz\right\}.\tag{12}$$

The relationships for velocity and pressure are not provided here in view of their size. It should be noted that in this approximation expressions for temperature and pressure do not depend on the form of $\sigma = \sigma(T)$.

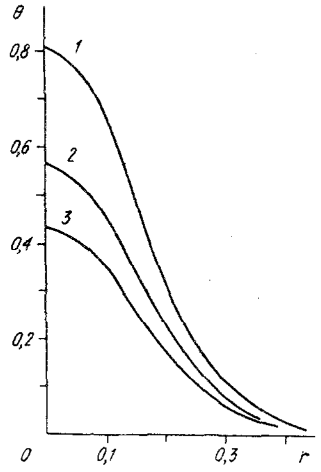

Fig. 1

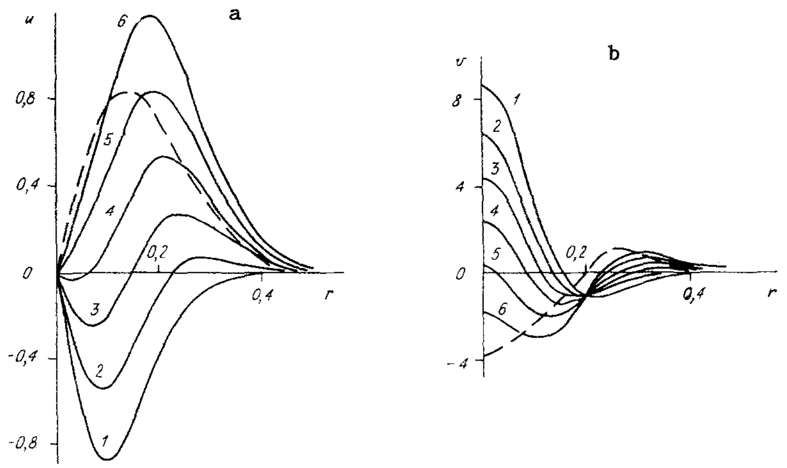

Fig. 2

The series obtained are summed on a computer. Curves are provided for a = 0.2, $\alpha H = 2$, Bd = 1, and Pr = 1. Shown in Fig. 1 is the temperature distribution over the radius at the liquid surface at different instants of time. Curves 1-3 correspond to times 0.01, 0.1, 1.0. As follows from (12), for time $t \geq 1$ temperature distribution does not depend on the coordinate z, and consequently curve 3 relates to temperature not only at the surface, but also for any z with $t \geq 1$. This limiting expression will be used as an initial condition for temperature with long times.

Presented in Fig. 2 is the distribution of component velocities at instant of time t = 0.1 for different values of θ^*: a is radial velocity at the liquid layer surface, b is vertical velocity at depth z = 0.95. With short times the liquid surface is still undeformed and the vertical velocity component at the surface equals zero. Curves 1-6 correspond to $\theta^* = 0, 0.2, 0.3, 0.4, 0.6, 0.8, 1$, and here and subsequently a broken line relates to the linear relationship $\sigma = \sigma_2(T')$.

As has been assumed, for all values of θ^* liquid movement is localized in a small region adjacent to the hot spot. The direction of liquid movement is determined by the direction of TC force operation, i.e., the sign of the gradient $d\sigma_1/dT' = 2\alpha_1(T' - T^*)$. With $\theta^* = 1$, $d\sigma_1/dT' < 0$ the qualitative picture of velocity distribution is similar to the linear case but with a slower increase close to the center of the vessel. With $\theta^* = 0$ within the volume spiral movement arises in the anticlockwise direction. For intermediate values of θ^* radial velocity takes both positive and negative values depending on the sign of $d\sigma_1/dT'$.

It is noted that with a change in sign of $d\sigma_1/dT'$ there is a delay in the change of direction for liquid movement in the surface layer. In view of the finite reserve of heat the temperature at the surface falls rapidly with time. At instant of time t = 0.1 at the sur-

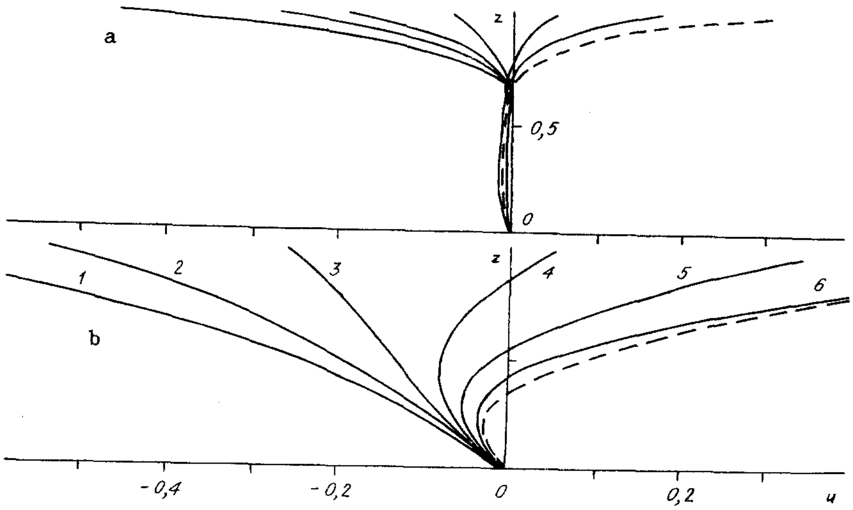

Fig. 3

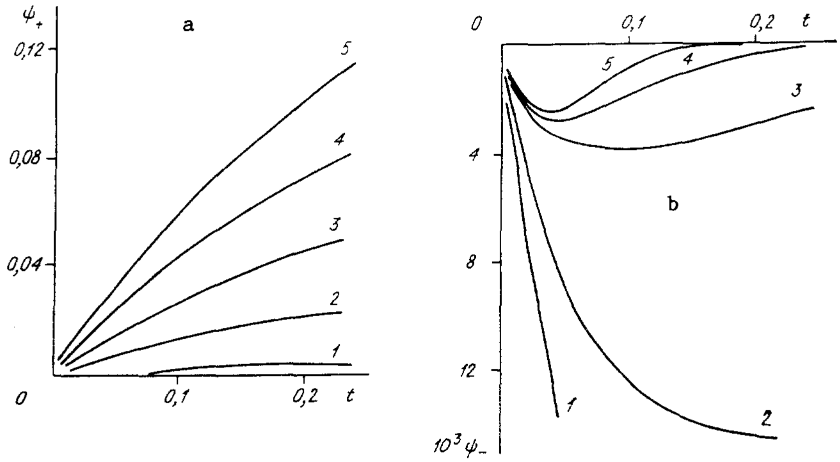

Fig. 4

face there is already no temperature $\theta^* \geq 0.6$ (Fig. 1, line 2), but in the liquid movement is maintained towards the center of the vessel (Fig. 2a, curve 4) in a small region close to the cell axis (Fig. 2b, curves 4 and 5 show movement upwards). This is connected with the inertia of volumetric spiral movement.

In Fig. 3, where the distribution is given for radial velocity over the height of the layer close to the axis ($r = 0.08$, on axis $u \equiv 0$), it is possible to follow development of movement within the layer (a is radial velocity for the initial instant of time $t = 0.01$, b is for $t = 0.1$, curves 1-6 relate to $\theta^* = 0$, 0.2, 0.4, 0.6, 0.8, 1).

As follows from Fig. 3a, the velocity close to the surface is much greater than within the volume. This means that in the initial instant of time the effect of TC forces which govern surface movement is greater than the effect of TG forces responsible for the velocity field within the layer. With time TC movement spreads over the whole layer, and the contribution of TG convection does not change significantly.

The maximum value of the flow function ψ_{max} specifies the intensity of movement. In the case of the nonlinear relationship $\sigma = \sigma_1(T')$ it may take both positive and negative values which correspond to the change in direction of TC force operation. Shown in Fig. 4 is the dependence of ψ_{max} on time for $\theta^* = 0.2$, 0.4, 0.6, 0.8, 1 (curves 1-5). For small θ^* (Fig. 4b) the intensity of "anomalous" spiral movement increases rapidly — it reaches a maximum more rapidly the greater θ^* — and then as high temperatures disappear from the system ($\theta > \theta^*$) with a delay it decreases to zero. In an ever spreading region, where the relationships $\theta < \theta^*$ are always fulfilled (the normal direction of operation of TC forces, ψ_+), with the short times in question the intensity of movement with time increases and emerges into a steady state (Fig. 4a).

Case of Long Times. With $\tau = \tau_2$ nonstationary terms in Eqs. (6) have the order ε^2 and they should be ignored. Here an equation is retained for the surface shape and the set of Eqs. (6)-(9) takes the form

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial r} + \frac{\mathrm{Bd}}{\beta_T}\frac{\partial h}{\partial r},$$
$$\frac{\partial p}{\partial z} = \mathrm{Bd}\,\theta,\ \frac{\partial^2 \theta}{\partial z^2} = 0,\ \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0. \tag{13}$$

At the free surface we have

$$p = 0,\ \frac{\partial u}{\partial z} = 2(\theta - \theta^*)\frac{\partial \theta}{\partial r},\ \frac{\partial \theta}{\partial z} = 0,\ \frac{\partial h}{\partial t} = v - u\frac{\partial h}{\partial r},$$
$$h = 1,\ t = 0,\ \frac{\partial h}{\partial r} = 0,\ r = 0;\ h = 1,\ \frac{\partial h}{\partial r} = 0,\ r = 1. \tag{14}$$

Boundary conditions at the bottom ($z = 0$) and with $r \to 1$ remain the same as for short times.

By solving Eq. (8) for temperature, from (8) and (13) we obtain

$$\theta = \theta(r,\ t),\ \partial\theta/\partial r = 0,\ r \to 1. \tag{15}$$

Whence it follows that a uniform temperature distribution is established over z, and this agrees with the conclusions made in the case of short times with $t \to \infty$ (12). The limiting solution of (12) for short times relates to (15) with $h(r,\ 0) = 1$. Naturally for the form $\theta(r,\ h(r,\ t))$ a limiting expression is taken

$$\theta(r,\ t) = \exp\left(-\frac{r^2}{a^2}\right)\frac{1 - \exp(-\alpha H h(r,\ t))}{\alpha H h(r,\ t)}.$$

Then by solving (13) using Eqs. (8) and (14) we find that

$$p = \mathrm{Bd}\,\theta(z - h),$$
$$v = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)\left\{z^2(\theta - \theta^*) + \mathrm{Bd}\left(\frac{z^4}{24} - \frac{hz^3}{6} + \frac{h^2 z^2}{4}\right)\right\} +$$
$$+ \frac{\mathrm{Bd}}{\beta_T}(1 - \beta_T\theta)\frac{(\dot h)^2 z^2}{2} + \mathrm{Bd}\left(\frac{z^3}{6} - \frac{hz^2}{2}\right)\left\{\dot h\frac{\partial\theta}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r\dot h}{\beta_T}(1 - \beta_T\theta)\right]\right\} - z^2\left(\frac{\partial\theta}{\partial r}\right)^2,$$
$$u = \frac{\partial\theta}{\partial r}\left(2z(\theta - \theta^*) + \mathrm{Bd}\left(\frac{z^3}{6} - \frac{hz^2}{2} + \frac{h^2 z}{4}\right)\right) +$$
$$+ \frac{\mathrm{Bd}}{\beta_T}(1 - \beta_T\theta)\dot h\left(\frac{z^2}{2} - hz\right).$$

For $h(r,\ t)$ an expression is obtained

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\frac{\partial\theta}{\partial r}\left(h^2(\theta^* - \theta) - \mathrm{Bd}\frac{h^4}{8}\right) + \frac{\mathrm{Bd}}{\beta_T}(1 - \beta_T\theta)\frac{\dot h h^3}{3}\right]\right\}. \tag{16}$$

We solve it assuming small surface deformations. We write $h(r,\ t) = 1 + \beta_T f(r,\ t)$, where $f \sim 1$, $\beta_T \sim 10^{-3}$. Here we assume that $\partial h/\partial t = \beta_T \partial f/\partial t \sim 1$, i.e., $\partial f/\partial t \sim 1/\beta_T$. Then

$$\theta \approx \exp\left(-\frac{r^2}{a^2}\right)\left\{\frac{1 - \exp(-\alpha H)}{\alpha H} + \beta_T f(r,\ t)\left(\exp(-\alpha H) - \right.\right.$$
$$\left.\left. - \frac{1 - \exp(-\alpha H)}{\alpha H}\right)\right\} = \exp\left(-\frac{r^2}{a^2}\right)(A + \beta_T f(r,\ t)B). \tag{17}$$

By substituting (17) in Eq. (16) and using expansion into a Taylor series with respect to β_T, we have

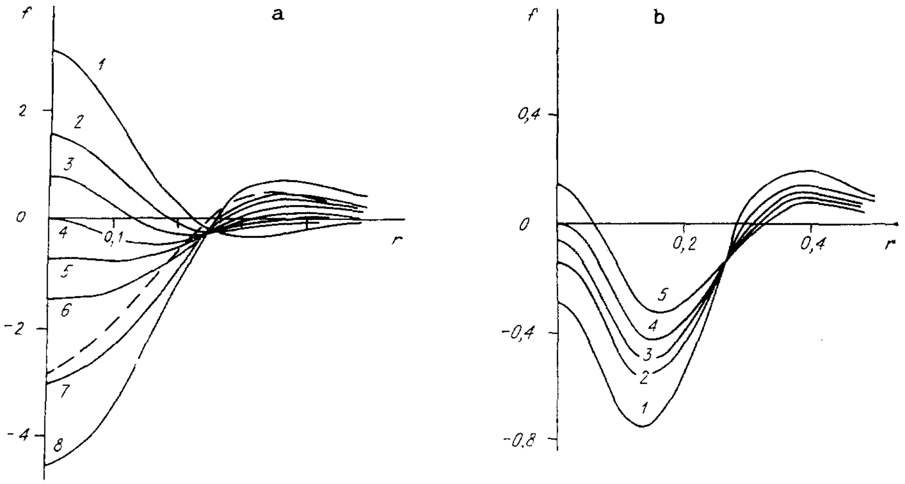

Fig. 5

$$\beta_T \frac{\partial f}{\partial t} = \frac{\text{Bd}}{3}\left(\ddot{f} + \frac{\dot{f}}{r}\right) + \frac{4A^2}{a^2}\left(1 - \frac{2r^2}{a^2}\right)\exp\left(-\frac{2r^2}{a^2}\right) - \\ - \frac{4A}{a^2}\left(1 - \frac{r^2}{a^2}\right)\exp\left(-\frac{r^2}{a^2}\right)\left(\theta^* - \frac{\text{Bd}}{8}\right), \\ f(t=0) = 0,\ \frac{\partial f}{\partial r}(r=0) = 0, \\ f(r\to\infty) = \frac{\partial f}{\partial r}(r\to\infty) = 0. \tag{18}$$

Solving (18) by means of zero-order Hankel transformations we find that

$$h = 1 + \frac{3}{8}\frac{\beta_T}{\text{Bd}}\frac{1-\exp(-\alpha H)}{\alpha H}\left\{(\text{Bd} - 8\theta^*)\left[\exp\left(-\frac{r^2}{a^2}\right) - \right.\right. \\ \left. - \exp\left(-\frac{r^2}{a^2}\left(1 + \frac{4}{3}\frac{\text{Bd}}{\beta_T}\frac{t}{a^2}\right)^{-1}\right)\left(1 + \frac{4\,\text{Bd}\,t}{3\beta_T a^2}\right)^{-1}\right] + \\ + 4\frac{1-\exp(-\alpha H)}{\alpha H}\left[\exp\left(-\frac{2r^2}{a^2}\right) - \right. \\ \left.\left. - \exp\left(-\frac{2r^2}{a^2}\left(1 + \frac{8\,\text{Bd}\,t}{3\beta_T a^2}\right)^{-1}\right)\left(1 + \frac{8\,\text{Bd}\,t}{\beta_T a^2}\right)^{-1}\right]\right\}.$$

Shown in Fig. 5a is surface shape at instant of time $t = 5 \cdot 10^{-4}$ for different θ^*, the broken line corresponds to the linear case, and curves 1-8 relate to $\theta^* = 0$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1. Curves are plotted for $a = 0.2$, $\alpha H = 2$, Bd = 1, $\beta_T = 10^{-3}$. With small changes in these parameters the substance of the picture is unchanged. There exist those θ^* (in our case $\theta^* \leq 0.3$) when close to the axis the surface protrudes. For $\theta^* = 0.5$ (curve 5) the surface with $r \to 0$ is convex, and with an increase in distance from the axis deflection develops and then again there is curvature. In the linear case close to the axis only deflection is observed, which corresponds to the experimental observations in [9].

Shown in Fig. 5b is the change in surface shape in relation to Bond number ($\theta^* = 0.4$, $t = 5 \cdot 10^{-4}$). Since in the theoretical consideration it was assumed that Bd ~ 1, the Bond number changes within small limits (0.7 ≤ Bd ≤ 1.3). Curves 1-5 relate to Bd = 0.7, 0.9, 1, 1.1, 1.3. It can be seen that with an increase in Bd, i.e., with an increase in the role of TC convection, there is an increase in surface shape deviation from flat.

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ANALOG OF THE SHALLOW-WATER VORTEX EQUATION FOR HOLLOW AND TORNADO-LIKE VORTICES.

HEIGHT OF A STEADY TORNADO-LIKE VORTEX

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UDC 532.5

The analog of the shallow-water vortex equation for hollow and tornado-like vortices is obtained in the long-wave approximation for an inviscid, incompressible, and nonuniform fluid. A steady, vertical tornado-like vortex is examined, whose central fluid is lighter than that outside the center. A sharp criterion is obtained, which distinguishes the case where the flow is bounded or unbounded in height. Calculation of the vortex height according to theoretical formula agrees in order of magnitude with the results of laboratory measurements and observations of naturally occurring dust devils.

1. Let us consider an incompressible, inviscid, nonuniform fluid in a gravitational field. The flow is assumed to be rotationally symmetric. We introduce a cylindrical coordinate system (r, φ, z), where r is the radius, and φ is the aximuthal angle. The z axis is directed opposite the force of gravity. The flow is divided into two regions in space: in region I, $r \le r_0(z, t)$; in region II, $r_0(z, t) \le r \le r_*$. Here r_* is a constant, r_0 is in general a function of z and t, and t is the time. At the boundary r_0, there can be a discontinuity in density and the component of velocity tangential to this boundary. The velocity components corresponding to (r, φ, z) are denoted by (u, v, w), and p, ρ, g are the pressure, density, and acceleration of gravity, respectively.

In order to change over to the long-wave approximation, subsequently we introduce characteristic length, velocity, and density scales. As the unit of length, we adopt the characteristic scale of change along the z axis and for unit velocity, the magnitude of the rotational component for $r = r_0$, $z = 0$, $t = 0$. The characteristic density is set equal to 1. Then the characteristic time, pressure, and acceleration are equal to 1. The characteristic scale for change along the r axis is denoted by δ. It is assumed that $\delta \ll 1$.

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 47-52, March-April, 1992. Original article submitted December 3, 1990.

0021-8944/92/3302-0185$12.50